Final Exam — Ordinary Differential Equations (WIGDV–07)

Wednesday 31 October 2018, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

Solve the following initial value problem:

$$y' = (x - y + 3)^2, \quad y(0) = 3.$$

Problem 2 (2 + 8 + 5 = 15 points)

Consider the following differential equation:

$$(x + y)\, dx - \frac{1}{x + y}\, dy = 0.$$

(a) Show that the equation is *not* exact.

(b) Compute an integrating factor of the form $M(x, y) = \phi(x + y)$.

(c) Compute the general solution in implicit form.

Problem 3 (3 + 12 + 5 = 20 points)

Consider the following 3×3 matrix:

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 3 & 0 \\ 1 & -2 & 4 \end{bmatrix}.$$

(a) Show that $\det(A - \lambda I) = (3 - \lambda)^3$.

(b) Compute the matrix J of the Jordan canonical form of A. (Do not compute Q!)

(c) Compute e^{Jt}.

Problem 4 (4 + 9 + 3 + 4 = 20 points)

Let $b > 0$ be arbitrary. The space $C([0,b]) = \{y : [0,b] \to \mathbb{R} : y \text{ is continuous}\}$ provided with the norm

$$\|y\| = \sup_{x\in[0,b]} |y(x)|e^{-2x}$$

is a Banach space. Consider the integral operator

$$T : C([0,b]) \to C([0,b]), \quad (Ty)(x) = \int_0^x \sin(e^t - y(t))\,dt.$$

Prove the following statements:

(a) $|\sin(e^x - y) - \sin(e^x - z)| \leq |y - z|$ for all $x, y, z \in \mathbb{R}$;

(b) $|(Ty)(x) - (Tz)(x)| \leq \frac{1}{2}(e^{2x} - 1)\|y - z\|$ for all $y, z \in C([0,b]),\ x \in [0,b]$;

(c) $\|Ty - Tz\| \leq \frac{1}{2}\|y - z\|$ for all $y, z \in C([0,b])$;

(d) The initial value problem

$$y' = \sin(e^x - y), \quad y(0) = 0,$$

has a unique solution on the interval $[0,b]$.

Problem 5 (10 points)

Compute the general solution of the following 4th order equation:

$$u'''' - 10u'' + 9u = 18 + 16e^t.$$

Note: a prime denotes differentiation with respect to t.

Problem 6 (3 + 6 + 6 = 15 points)

Consider the semi-homogeneous boundary value problem

$$u'' = \sin(\pi x), \qquad u'(0) = 0, \qquad u(1) = 0,$$

(a) Solve the boundary value problem directly *without* using Green's function.

(b) Compute Green's function.

(c) Solve the boundary value problem using Green's function.

End of test (90 points)

Solution of problem 1 (10 points)

Method 1: using a substitution. Let $u = x - y + 3$, then

$$u' = 1 - y' = 1 - (x - y + 3)^2 = 1 - u^2.$$

Separation of variables gives

$$\int \frac{1}{1-u^2}\, du = \int dx.$$

(1 point)

The left hand integral can be computed using a partial fraction expansion:

$$\int \frac{1}{1-u^2} = \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\, du = \frac{1}{2}\big(\log|1+u| - \log|1-u|\big) = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|.$$

(4 points)

Hence, we obtain the solution

$$\log\left|\frac{1+u}{1-u}\right| = 2x + C \quad \Rightarrow \quad \frac{1+u}{1-u} = Ke^{2x} \quad (K = \pm e^C).$$

Solving for u, and next for y, gives

$$u = \frac{Ke^{2x}-1}{Ke^{2x}+1} \quad \Rightarrow \quad y = x + 3 - u = x + 3 - \frac{Ke^{2x}-1}{Ke^{2x}+1}.$$

(3 points)

Finally, the initial condition $y(0) = 3$ gives

$$3 - \frac{K-1}{k+1} = 3 \quad \Rightarrow \quad K = 1.$$

Therefore, the solution of the initial value problem is given by

$$y = x + 3 - \frac{e^{2x}-1}{e^{2x}+1}.$$

(2 points)

Method 2: solving as a Riccati equation (not recommended). Expanding brackets and rearranging terms gives the following Riccati equation:

$$y' = -(2x+6)y + y^2 + x^2 + 6x + 9.$$

A particular solution is given by $\phi(x) = x + 2$. (This is best seen by guessing a solution *before* expanding brackets.)
(2 points)

Setting $u = y - \phi$ gives the following Bernoulli equation:

$$u' = -2u + u^2.$$

(2 points)

Setting $z = 1/u$ gives the following linear equation:

$$z' = 2z - 1.$$

(2 points)

Solving for z gives

$$z' - 2z = -1 \quad \Rightarrow \quad [e^{-2x}z]' = -e^{-2x} \quad \Rightarrow \quad z = \tfrac{1}{2} + Ce^{2x}.$$

(2 points)

Therefore, we get

$$y = u + x + 2 = \frac{1}{z} + x + 2 = \frac{1}{Ce^{2x} + \frac{1}{2}} + x + 2.$$

The initial condition $y(0) = 3$ gives $C = \frac{1}{2}$.
(2 points)

Method 3: solving as a Riccati equation (not recommended). Expanding brackets and rearranging terms gives the following Riccati equation:

$$y' = -(2x+6)y + y^2 + x^2 + 6x + 9.$$

We can also take the particular solution $\phi(x) = x + 4$.
(2 points)

Setting $u = y - \phi$ gives the following Bernoulli equation:

$$u' = 2u + u^2.$$

(2 points)

Setting $z = 1/u$ gives the linear equation

$$z' = -2z - 1.$$

(2 points)

Solving for z gives

$$z' + 2z = -1 \quad \Rightarrow \quad [e^{2x}z]' = -e^{2x} \quad \Rightarrow \quad z = -\tfrac{1}{2} + Ce^{-2x}.$$

(2 points)

Therefore, we get

$$y = u + x + 4 = \frac{1}{z} + x + 4 = \frac{1}{Ce^{-2x} - \frac{1}{2}} + x + 4.$$

The initial condition $y(0) = 3$ gives $C = -\frac{1}{2}$.
(2 points)

Solution of problem 2 (2 + 8 + 5 = 15 points)

(a) We can write the equation as $g(x, y)\,dx + h(x, y)\,dy = 0$ where $g(x, y) = x + y$ and $h(x, y) = -1/(x+y)$. The equation is called exact when $g_y = h_x$. However, we have that

$$g_y = 1 \quad \text{and} \quad h_x = \frac{1}{(x+y)^2}.$$

Therefore, the equation is not exact.
(2 points)

(b) Setting $M(x, y) = \phi(x + y)$ gives

$$(Mg)_y = \phi(x+y) + (x+y)\phi'(x+y),$$
$$(Mh)_x = \frac{1}{(x+y)^2}\phi(x+y) - \frac{1}{x+y}\phi'(x+y).$$

(2 points)

The function M is an integrating factor if and only if $(Mg)_y = (Mh)_x$. By setting $t = x + y$ we can rewrite the differential equation for ϕ as follows:

$$\left(t + \frac{1}{t}\right)\phi'(t) + \left(1 - \frac{1}{t^2}\right)\phi(t) = 0,$$

or, equivalently,

$$\frac{d}{dt}\left[\left(t + \frac{1}{t}\right)\phi(t)\right] = 0.$$

(3 points)

The general solution is given by

$$\phi(t) = \frac{C}{t + 1/t} = \frac{Ct}{1+t^2},$$

where C is an arbitrary constant. Taking $C = 1$ gives the following integrating factor:

$$M(x, y) = \frac{x+y}{1 + (x+y)^2}.$$

(3 points)

Alternative approach: We can also write the equation for ϕ as follows:

$$\phi'(t) = \frac{1-t^2}{t(1+t^2)}\phi(t).$$

Applying partial fraction expansion gives

$$\frac{1-t^2}{t(1+t^2)} = \frac{A}{t} + \frac{Bt + C}{1+t^2},$$

or, equivalently,

$$1 - t^2 = A(1+t^2) + (Bt + C)t.$$

Comparing like powers of t gives the coefficients $A = 1$, $B = -2$, and $C = 0$. Solving the differential equation then gives

$$\begin{aligned}\phi(t) &= \exp\left(\int \frac{1-t^2}{t(1+t^2)}\,dt\right)\\ &= \exp\left(\int \frac{1}{t} - \frac{2t}{1+t^2}\,dt\right)\\ &= \exp\left(\ln|t| - \ln|1+t^2| + C\right)\\ &= \exp\left(\ln\left|\frac{t}{1+t^2}\right| + C\right)\\ &= K\left|\frac{t}{1+t^2}\right|.\end{aligned}$$

Note that we may leave out the absolute value bars by absorbing a minus sign into K. Hence, we get as a possible solution the function

$$\phi(t) = \frac{t}{1+t^2}.$$

(c) After multiplication with the integrating factor we can rewrite the given differential as follows:

$$\left(1 - \frac{1}{1+(x+y)^2}\right)dx - \frac{1}{1+(x+y)^2}\,dy = 0.$$

Define the function

$$F(x,y) = \int 1 - \frac{1}{1+(x+y)^2}\,dx = x - \arctan(x+y) + C(y).$$

(2 points)

Differentiating with respect to the y variables gives

$$F_y = -\frac{1}{1+(x+y)^2} + C'(y) \quad \Rightarrow \quad C'(y) = 0.$$

By choosing $C(y) = 0$ we obtain the following potential function:

$$F(x,y) = x - \arctan(x+y),$$

(2 points)

Hence, the general solution of the differential equation is given by the following implicit equation:

$$x - \arctan(x+y) = K,$$

where $K \in \mathbb{R}$ is an arbitrary constant.
(1 point)

Solution of problem 3 (3 + 12 + 5 = 20 points)

(a) Expanding the determinant along the *second* row gives:

$$\begin{aligned}\det(A-\lambda I) &= \begin{bmatrix} 2-\lambda & 2 & -1 \\ 0 & 3-\lambda & 0 \\ 1 & -2 & 4-\lambda \end{bmatrix} \\ &= (3-\lambda)\det\begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} \\ &= (3-\lambda)((2-\lambda)(4-\lambda)+1) \\ &= (3-\lambda)(\lambda^2-6\lambda+9) \\ &= (3-\lambda)(\lambda-3)^2 \\ &= (3-\lambda)^3. \end{aligned}$$

(3 points)

(b) The generalized eigenspaces of A for $\lambda = 3$ are given by

$$A-\lambda I = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^1 = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(4 points)

$$(A-\lambda I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(4 points)

Therefore, the dot diagram is given by

$$\left.\begin{aligned} r_1 &= \dim E_\lambda^1 = 2 \\ r_2 &= \dim E_\lambda^2 - \dim E_\lambda^1 = 1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{matrix} \bullet & \bullet \\ \bullet & \end{matrix}$$

(2 points)

This means that there is one cycle of length 2 and one cycle of length 1. In particular, the matrix J is given by

$$J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(2 points)

(c) In the first case, we can write

$$J = D + N \quad \text{where} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $DN = ND$ we have $e^{Jt} = e^{Dt+Nt} = e^{Dt}e^{Nt}$. The series expansion for the exponential function gives

$$e^{Nt} = I + Nt + N^2t^2 + \cdots = I + Nt = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(3 points)

which implies that

$$e^{Jt} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} & te^{3t} & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

(2 points)

Solution of problem 4 (4 + 9 + 3 + 4 = 20 points)

(a) Set $u = e^x - y$ and $v = e^x - z$. If $u = v$, then there is nothing to prove. If $u \neq v$, then the Mean Value Theorem asserts the existence of a real number w between u and v such that

$$\sin(u) - \sin(v) = \cos(w)(u - v).$$

(2 points)

Taking absolute values gives

$$\begin{aligned} |\sin(e^x - y) - \sin(e^x - z)| &= |\cos(w)|\,|(e^x - y) - (e^x - z)| \\ &= |\cos(w)|\,|y - z| \\ &\leq |y - z|. \end{aligned}$$

(2 points)

(b) Assume that $y, z \in C([0, b])$. For each $x \in [0, b]$ we have

$$\begin{aligned} |(Ty)(x) - (Tz)(x)| &= \left| \int_0^x \sin(e^t - y(t)) - \sin(e^t - z(t))\, dt \right| \\ &\leq \int_0^x |\sin(e^t - y(t)) - \sin(e^t - z(t))|\, dt \\ &\leq \int_0^x |y(t) - z(t)|\, dt \\ &= \int_0^x |y(t) - z(t)| e^{-2t} e^{2t}\, dt \\ &\leq \|y - z\| \int_0^x e^{2t}\, dt \\ &= \tfrac{1}{2}(e^{2x} - 1)\|y - z\|. \end{aligned}$$

(9 points)

(c) From part (b) we obtain the inequality for all $x \in [0, b]$:

$$|(Ty)(x) - (Tz)(x)|e^{-2x} \leq \tfrac{1}{2}(1 - e^{-2x})\|y - z\| \leq \tfrac{1}{2}\|y - z\|.$$

Now taking the supremum over all $x \in [0, b]$ gives the desired result.
(3 points)

(d) The initial value problem

$$y' = \sin(e^x - y), \quad y(0) = 0,$$

is equivalent to the fixed point equation $Ty = y$. According to the Banach fixed point theorem the latter equation has a unique solution in the Banach space $C([0, b])$.
(4 points)

Solution of problem 5 (10 points)

This is a linear, non-homogeneous differential equation of 4th order with constant coefficients. We first solve the homogeneous equation

$$u'''' - 10u'' + 9u = 0$$

by setting $u(t) = e^{\lambda t}$. This gives the following characteristic equation:

$$\lambda^4 - 10\lambda^2 + 9 = 0.$$

(1 point)

Solving the characteristic equation gives

$$\begin{aligned} \lambda^4 - 10\lambda^2 + 9 = 0 \quad &\Leftrightarrow \quad (\lambda^2 - 1)(\lambda^2 - 9) = 0 \\ &\Leftrightarrow \quad (\lambda - 1)(\lambda + 1)(\lambda - 3)(\lambda + 3) = 0 \\ &\Leftrightarrow \quad \lambda = 1 \text{ or } \lambda = 3 \text{ or } \lambda = -3. \end{aligned}$$

(3 points)

To find a particular solution to the non-homogeneous equation we apply the method of undetermined coefficients with the educated guess $u_p(t) = a + bte^t$. We have

$$u_p' = b(t+1)e^t, \quad u_p'' = b(t+2)e^t, \quad u_p''' = b(t+3)e^t, \quad u_p'''' = b(t+4)e^t.$$

Therefore,

$$u_p'''' - 10u_p'' + 9u_p = 9a - 16be^t,$$

which implies that we have to take $a = 2$ and $b = -1$.
(5 points)

Finally, the general solution is given by

$$u(t) = c_1e^t + c_2e^{-t} + c_3e^{3t} + c_4e^{-3t} + 2 - te^t,$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are arbitrary constants.
(1 point)

Solution of problem 6 (3 + 6 + 6 = 15 points)

(a) Integrating the differential equation twice gives

$$u(x) = ax + b - \frac{1}{\pi^2}\sin(\pi x).$$

The boundary values of this function are

$$u'(0) = a - \frac{1}{\pi} \quad \text{and} \quad u(1) = a + b.$$

The boundary conditions imply that $a = 1/\pi$ and $b = -1/\pi$.
(3 points)

(b) The general solution of the homogeneous differential equation $u'' = 0$ is given by $u(x) = ax + b$.
(1 point)

The function $u_1(x) = 1$ satisfies the boundary condition $u'(0) = 0$, and the function $u_2(x) = x - 1$ satisfies the boundary condition $u(1) = 0$.
(2 points)

The Wronskian determinant is given by

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) = 1.$$

(1 point)

Hence, Green's function is given by

$$\Gamma(x,\xi) = \begin{cases} x - 1 & \text{if } 0 \le \xi \le x \le 1, \\ \xi - 1 & \text{if } 0 \le x \le \xi \le 1. \end{cases}$$

(2 points)

(c) Using Green's function, the solution of the boundary value problem is given by

$$\begin{aligned}
u(x) &= \int_0^1 \Gamma(x,\xi) f(\xi)\, d\xi \\
&= \int_0^x \Gamma(x,\xi) f(\xi)\, d\xi + \int_x^1 \Gamma(x,\xi) f(\xi)\, d\xi \\
&= (x-1)\int_0^x \sin(\pi\xi)\, d\xi + \int_x^1 (\xi - 1)\sin(\pi\xi)\, d\xi \\
&= (x-1)\left[-\frac{\cos(\pi\xi)}{\pi}\right]_0^x + \left[-\frac{\xi-1}{\pi}\cos(\pi\xi)\right]_x^1 + \int_x^1 \frac{1}{\pi}\cos(\pi\xi)\, d\xi \\
&= (x-1)\left[-\frac{\cos(\pi\xi)}{\pi}\right]_0^x + \left[-\frac{\xi-1}{\pi}\cos(\pi\xi)\right]_x^1 + \left[\frac{1}{\pi^2}\sin(\pi\xi)\right]_x^1 \\
&= -\frac{x-1}{\pi}\cos(\pi x) + \frac{x-1}{\pi} + \frac{x-1}{\pi}\cos(\pi x) - \frac{1}{\pi^2}\sin(\pi x) \\
&= \frac{x-1}{\pi} - \frac{1}{\pi^2}\sin(\pi x).
\end{aligned}$$

(6 points)